

INVOLUTORIAL SPACE TRANSFORMATIONS ASSOCIATED
WITH A LINEAR CONGRUENCE

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INVOLUTORIAL SPACE TRANSFORMATIONS ASSOCIATED WITH A LINEAR CONGRUENCE

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1. **Introduction.** In a paper [2] published some years ago, Vogt studied the space transformations associated with a linear congruence, using synthetic methods. Recently the present author [1] gave an analytical discussion of the non-involutorial transformations associated with these configurations, pointing out some properties overlooked by Vogt's study. The involutorial transformations involved are dismissed by Vogt, in the paper mentioned above, with little more than mere mention. In the present paper these transformations are more fully discussed, chiefly by the use of analysis. A number of interesting properties are thus disclosed, among these being the existence of a rather large number of parasitic lines.

Given the directrices r and s of the congruence and a pencil of surfaces

$$|F_{m+n+2}|: r^m s^n g_{2mn+4m+4n+4}.$$

Through a generic point $P(y)$ there passes a single F of $|F|$. The unique line t of the congruence through $P(y)$ meets F a second time in one residual point $Q(x)$, the image of $P(y)$ under the transformation thus defined. The residual base curve of $|F|$ has been denoted by g . As indicated above, the surfaces of the pencil are of order $m+n+2$, r and s lying m times and n times, respectively, on each, while g is of order $2mn+4m+4n+4$. It will be shown that r , s , and g are fundamental curves of the involution.

2. **Equations of the transformation.** Let us take the equations of the directrices r and s , respectively, as

$$(1) \quad x_1 = x_2 = 0, \quad x_3 = x_4 = 0,$$

and the pencils of surfaces $|F|$ as

$$(2) \quad |F_{m+n+2}| \equiv U - \lambda U' = 0,$$

where

$$(3) \quad \begin{aligned} U_{m+n+2} &= \sum_{i,k=0}^{m,n} u_i v_k (c_{ik} x), & U'_{m+n+2} &= \sum_{i,k=0}^{m,n} u'_i v'_k (c'_{ik} x), \\ u_i &= \sum_{j=0}^m a_{ij} x_1^i x_2^{m-i}, & u'_i &= \sum_{j=0}^m a'_{ij} x_1^i x_2^{m-i}, \\ v_k &= \sum_{j=0}^n b_{kj} x_3^k x_4^{n-k}, & v'_k &= \sum_{j=0}^n b'_{kj} x_3^k x_4^{n-k}, \\ (c_{ik} x) &= \sum_{p,q=1}^4 c_{ik,pq} x_p x_q, \end{aligned}$$

and so on.

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Through a generic point $P(y)$ there is an F of $|F|$ with parameter $\lambda = U(y)/U'(y)$ and equation

$$(4) \quad U(x)U'(y) - U(y)U'(x) = 0.$$

The unique transversal t through $P(y)$, r , and s meets the surface (4) a second time in a point $Q(x)$ having coordinates

$$(5) \quad I_{2m+2n+5} : \left\{ \begin{array}{l} \sigma x_1 = Ry_1, \\ \sigma x_2 = Ry_2, \\ \sigma x_3 = (R - K)y_3 = Sy_3, \\ \sigma x_4 = (R - K)y_4 = Sy_4, \end{array} \right.$$

where

$$(6) \quad \begin{aligned} R_{2m+2n+4} &= U'W - UW', & S_{2m+2n+4} &= U'V - UV', \\ K_{2m+2n+4} &= U'(W - V) - U(W' - V'), \\ V_{m+n+2} &= \sum_{j,k=0}^{m,n} u_j v_k [c_{12}y], & V'_{m+n+2} &= \sum_{j,k=0}^{m,n} u'_j v'_k [c'_{12}y], \\ W_{m+n+2} &= \sum_{j,k=0}^{m,n} u_j v_k [c_{34}y], & W'_{m+n+2} &= \sum_{j,k=0}^{m,n} u'_j v'_k [c'_{34}y], \\ [c_{12}y] &= c_{jk,11}y_1^2 + 2c_{jk,12}y_1y_2 + c_{jk,22}y_2^2, \\ [c_{34}y] &= c_{jk,33}y_3^2 + 2c_{jk,34}y_3y_4 + c_{jk,44}y_4^2, \end{aligned}$$

and so on.

Equations (5) are those of the involution under consideration.

3. Images of fundamental curves. The transformation I applied to an F of $|F|$ gives

$$(7) \quad U \sim (I)UR^mS^nG,$$

where

$$G_{4m+4n+8} = RS + K(V'W - VW').$$

Here $r \sim (I)R$, $s \sim (I)S$, $g \sim (I)G$. Similarly

$$(8) \quad \begin{aligned} R &\sim (I)R^{2m+1}S^{2n+2}G, & S &\sim (I)R^{2m+2}S^{2n+1}G, \\ G &\sim (I)R^{4m+4}S^{4n+4}G, & K &\sim (I)KR^{2m+1}S^{2n+1}G. \end{aligned}$$

Through a point O_r on r there is a pencil of transversals through s . Each direction in this pencil determines an F which cuts the line in one residual point. The locus of such points is a plane curve c which generates the surface

R , image of r . The order of c , determined by the intersection of R and a homaloidal surface, is $2m + 2$. Similarly, taking a point O_s on s , we find a plane curve k of order $2n + 2$ generating the surface S , image of s .

Through a point O_g on g there is a unique line t of the congruence. However, every F of $|F|$ contains O_g , hence $O_g \sim (I)t$. The ruled surface G generated by t is the image of g under I .

Let us designate a point common to r and g as O_{rg} . The image of such a point, since it lies on r , is a c_{2m+2} . However, since the point also lies on g , the c_{2m+2} must contain a line l , image of g , hence is composite. The number of such lines l , hence the number of points of intersection of r and g , is shown by the intersection of R and G to be $2mn + 4m$. In an analogous fashion, there are $2mn + 4n$ points common to s and g , hence $2mn + 4n$ lines q common to S and G .

4. Contact along r , s , and g . The equations of the directrix r may be written as $x_1 = 0, x_2 = 0, x_3 = k, x_4 = 1$. Let us set up a projectivity which will carry a generic point of r into a vertex of the coordinate tetrahedron, say in the manner

$$\begin{aligned}(1, 0, 0, 0) &\rightarrow (1, 0, 0, 0), & (0, 1, 0, 0) &\rightarrow (0, 1, 0, 0), \\ (0, 0, k, 1) &\rightarrow (0, 0, 1, 0), & (0, 0, 0, 1) &\rightarrow (0, 0, 0, 1).\end{aligned}$$

The required projectivity takes the form $y_1 = x_1, y_2 = x_2, y_3 = kx_3, y_4 = x_3 + x_4$.

When this projectivity is applied to the fundamental surfaces of the transformation, it is found that the coefficients of x_3 in the transformed equations of R and K have a common factor of order $2m + 1$ which is

$$\begin{aligned}&\sum_{j,k=0}^{m,n} u_j \sum_{i=0}^n b_{ik} k^i (c_{ik,33} k^2 + 2c_{ik,34} k + c_{ik,44}) \\&\quad \cdot \sum_{j,k=0}^{m,n} u'_j \sum_{i=0}^n b'_{ik} k^i [2(c'_{ik,13} k + c'_{ik,14}) y_1 + 2(c'_{ik,23} k + c'_{ik,24}) y_2] \\&\quad - \sum_{j,k=0}^{m,n} u'_j \sum_{i=0}^n b'_{ik} k^i (c'_{ik,33} k^2 + 2c'_{ik,34} k + c'_{ik,44}) \\&\quad \cdot \sum_{j,k=0}^{m,n} u_j \sum_{i=0}^n b_{ik} k^i [2(c_{ik,13} k + c_{ik,14}) y_1 + 2(c_{ik,23} k + c_{ik,24}) y_2].\end{aligned}$$

This is indicative of contact of order $2m + 1$ between R and K along r .

When a similar projectivity carrying an arbitrary point of the directrix s into the vertex $(1, 0, 0, 0)$ of the reference tetrahedron is set up and applied

to the fundamental surfaces, the coefficients of x_1 in the transformed equations of S and K are found to have the following common factor of order $2n + 1$:

$$\begin{aligned} & \sum_{i,k=0}^{m,n} v_k \sum_{i=0}^m a_{ij} k^i (c_{jk,11} k^2 + 2c_{jk,12} k + c_{jk,22}) \\ & \quad \cdot \sum_{i,k=0}^{m,n} v'_k \sum_{i=0}^m a'_{ij} k^i [2(c'_{jk,13} k + c'_{jk,23}) y_3 + 2(c'_{jk,14} k + c'_{jk,24}) y_4] \\ & - \sum_{i,k=0}^{m,n} v'_k \sum_{i=0}^m a'_{ij} k^i (c'_{jk,11} k^2 + 2c'_{jk,12} k + c'_{jk,22}) \\ & \quad \cdot \sum_{i,k=0}^{m,n} v_k \sum_{i=0}^m a_{ij} k^i [2(c_{jk,13} k + c_{jk,23}) y_3 + 2(c_{jk,14} k + c_{jk,24}) y_4]. \end{aligned}$$

It follows that S and K have contact of order $2n + 1$ along s .

To investigate the existence of tangency between fundamental surfaces along the base curve g , the fundamental system is cut by an arbitrary plane, Π , containing r (or s). Through each point of intersection, Γ_i , of Π and g there is one line, γ_i , of the congruence, which, since Γ_i lies on every F of $|F|$, is the image of Γ_i . It follows that $\Gamma_i \sim (I)\Gamma_i$ (along with the other points of γ_i), hence is a point on κ , the section of the invariant surface K . In general there is only one point Γ_i on γ_i , for otherwise γ_i would be a parasitic line of which there are only a finite number (see §5).

The line γ_i , however, cuts κ in $2m + 2n + 4 - (2m + 1) - (2n + 1) = 2$ points not on r or s . This second intersection cannot be distinct from Γ_i for every point on γ_i is carried into Γ_i by the involution, whereas every point on κ is invariant. Furthermore, Γ_i is not a double point on κ for we have found in (8) that $K \sim (I)KR^{2m+1}S^{2n+1}G$, showing that g is single, not double, on K . It follows that γ_i is tangent to κ at Γ_i , so that G and K have simple tangency along g .

5. Parasitic lines. Any line p of the congruence which meets g twice will be such that every point of the line is transformed into the entire line, i.e., is parasitic. Such a line is double on G and lies once upon R , S , K , and each ϕ . The number of parasitic lines, as determined by the intersection of two homaloidal surfaces, is $6mn + 6m + 6n + 8$.

6. Invariant and homaloidal surfaces. It is evident from (5) that the surface $K = 0$ is pointwise invariant under I .

A generic plane π when subjected to the involution gives

$$\pi = \sum_{i=1}^4 A_i x_i \sim (I)R(A_1 y_1 + A_2 y_2) + S(A_3 y_3 + A_4 y_4) = \phi_{2m+2n+5},$$

where the ϕ 's are homaloidal surfaces of the transformation. Further,

$$\phi \sim (I)R^{2m+2}S^{2n+2}G,$$

hence the homaloidal web is

$$\infty^3 \mid \phi \mid : r^{2m+2} s^{2n+2} g(6mn + 6m + 6n + 8)p.$$

The intersection of two homaloidal surfaces gives the homaloidal net

$$H \equiv [\phi\phi] : r^{4m^2+8m+4} s^{4n^2+8n+4} g(6mn + 6m + 6n + 8) p c_{2m+2n+5},$$

where $c_{2m+2n+5}$ is the image of the line $[\pi\pi]$.

The jacobian of the involution is $J_{8m+8n+16} = RSG$.

7. Table of characteristics. The images of planes and of the fundamental elements may now be expressed by the following table:

$$\pi \sim (I) \quad \phi : r^{2m+2} s^{2n+2} g(6mn + 6m + 6n + 8)p,$$

$$r \sim (I) \quad R : r^{2m+1+(2m+1)t} s^{2n+2} g(2mn + 4m)l(6mn + 6m + 6n + 8)p,$$

$$s \sim (I) \quad S : r^{2m+2} s^{2n+1+(2n+1)t} g(2mn + 4n)q(6mn + 6m + 6n + 8)p,$$

$$g \sim (I) \quad G : r^{4m+4} s^{4n+4} g^{1+1t} (2mn + 4m)l(2mn + 4n)q2(6mn + 6m + 6n + 8)p,$$

$$K \sim (I) \quad K : r^{2m+1+(2m+1)t} s^{2n+1+(2n+1)t} g^{1+1t} (6mn + 6m + 6n + 8)p,$$

where the coefficients of t in the multiplicities of r , s , and g indicate the order of contact as found in §4.

8. Intersection table. The complete intersection table may now be written as follows:

$$[\phi R] : r^{4m^2+6m+2} s^{4n^2+8n+4} g(6mn + 6m + 6n + 8) p c_{2m+2},$$

$$[\phi S] : r^{4m^2+8m+4} s^{4n^2+6n+2} g(6mn + 6m + 6n + 8) p k_{2n+2},$$

$$[\phi G] : r^{8m^2+16m+8} s^{8n^2+16n+8} g2(6mn + 6m + 6n + 8) p(2mn + 4m + 4n + 4)d,$$

$$[\phi K] : r^{4m^2+6m+2} s^{4n^2+6n+2} g(6mn + 6m + 6n + 8) p f_{2m+2n+4},$$

$$[RS] : r^{4m^2+6m+2} s^{4n^2+6n+2} g(6mn + 6m + 6n + 8)p,$$

$$[RG] : r^{8m^2+12m+4} s^{8n^2+16n+8} g(2mn + 4m)l2(6mn + 6m + 6n + 8)p,$$

$$[RK] : r^{4m^2+4m+1+(2m+1)t} s^{4n^2+6n+2} g(6mn + 6m + 6n + 8)p,$$

$$[SG] : r^{8m^2+16m+8} s^{8n^2+12n+4} g(2mn + 4n)q2(6mn + 6m + 6n + 8)p,$$

$$[SK] : r^{4m^2+6m+2} s^{4n^2+4n+1+(2n+1)t} g(6mn + 6m + 6n + 8)p,$$

$$[GK] : r^{8m^2+12m+4} s^{8n^2+12n+4} g^{1+1t} 2(6mn + 6m + 6n + 8)p.$$

The lines d are the images of the points of intersection of π and g , while $f_{2m+2n+4}$ is the intersection $[\pi K]$.

9. **Degeneracies.** Throughout this paper the base curve g is considered to be non-composite. It may be pointed out that if g should become composite in such a way as to give rise to a ruled surface P_δ of parasitic lines, quadrisecants of the system r , s , and g , the order of the involution would be reduced by δ , order of P , becoming $2m + 2n + 5 - \delta$.

10. **The I_{2n+3} in a plane through r .** A plane $\pi \equiv x_1 = \beta x_2$ cuts the surfaces of $|F|$ in residual pencils of curves

$$(9) \quad |f_{n+2}| \equiv u - \mu u' = 0,$$

where

$$\begin{aligned} u_{n+2} &= \sum_{j,k=0}^{m,n} \psi_j \omega_k (\alpha_{jk} x), & u'_{n+2} &= \sum_{j,k=0}^{m,n} \psi'_j \omega'_k (\alpha'_{jk} x), \\ \psi_j &= \sum_{i=0}^m a_{ij} \beta^i, & \psi'_j &= \sum_{i=0}^m a'_{ij} \beta^i, & \omega_k &= \sum_{i=0}^n b_{ik} x_3^i x_4^{n-i}, \\ (10) \quad \omega'_k &= \sum_{i=0}^n b'_{ik} x_3^i x_4^{n-i}, \\ (\alpha_{jk} x) &= (\alpha_{jk,11} \beta^2 + 2\alpha_{jk,12} \beta + \alpha_{jk,22}) x_2^2 \\ &\quad + 2[(\alpha_{jk,13} \beta + \alpha_{jk,23}) x_3 + (\alpha_{jk,14} \beta + \alpha_{jk,24}) x_4] x_2 \\ &\quad + (\alpha_{jk,33} x_3^2 + 2\alpha_{jk,34} x_3 x_4 + \alpha_{jk,44} x_4^2). \end{aligned}$$

The second directrix s cuts π in a point $S : (\beta, 1, 0, 0)$ which is a n -fold point on the curves of $|f|$. Through a generic point $P(y)$ of π passes an f of $|f|$ having parameter $\mu = u(y)/u'(y)$ and equation

$$(11) \quad u(x)u'(y) - u(y)u'(x) = 0.$$

The unique transversal t through $P(y)$ and S meets (11) again in one residual point $Q(x)$, image of $P(y)$ under the involution thus defined. The involution has as its equations

$$(12) \quad I_{2n+3} : \quad x_2 = \sigma y_2 + \kappa, \quad x_3 = \sigma y_3, \quad x_4 = \sigma y_4,$$

where

$$\begin{aligned} \sigma_{2n+2} &= u'v - uv', & \kappa_{2n+3} &= -2[\sigma y_2 + (u'w - uw')], \\ (13) \quad v_n &= \sum_{j,k=0}^{m,n} \psi_j \omega_k (\alpha_{jk,11} \beta^2 + 2\alpha_{jk,12} \beta + \alpha_{jk,22}), \\ w_{n+1} &= \sum_{j,k=0}^{m,n} \psi_j \omega_k [(\alpha_{jk,13} \beta + \alpha_{jk,23}) x_3 + (\alpha_{jk,14} \beta + \alpha_{jk,24}) x_4], \end{aligned}$$

and so on.

The curve κ is pointwise invariant under the involution and $S \sim (I)\sigma$. Upon applying the involution, we get

$$(14) \quad \begin{aligned} u &\sim (I)u\sigma^n\gamma, & \sigma &\sim (I)\sigma^{2n+1}\gamma, \\ \gamma &\sim (I)\sigma^{4n+4}\gamma, & \kappa &\sim (I) - \kappa\sigma^{2n+1}\gamma, \\ \gamma_{4n+4} &= \sigma^2 + 2\kappa(w'v - wv'). \end{aligned}$$

The base points of the pencil $|f|$ are S and $4n + 4$ other points $\Gamma_1, \dots, \Gamma_{4n+4}$. We designate the lines $S\Gamma_1, \dots, S\Gamma_{4n+4}$ as $\gamma_1, \dots, \gamma_{4n+4}$, respectively. The curve γ in (14) consists of the lines $\gamma_1, \dots, \gamma_{4n+4}$ and $\Gamma_i \sim (I)\gamma_i$.

When the involution is applied to a generic line, we get

$$\begin{aligned} \lambda &\equiv a_2x_2 + a_3x_3 + a_4x_4 \sim (I)\sigma\lambda + a_2\kappa = \phi_{2n+3}, \\ \phi &\sim (I)\lambda\sigma^{2n+2}\gamma \end{aligned}$$

and the homaloidal net is

$$\infty^2 |\phi| : S^{2n+2}\Gamma_1, \dots, \Gamma_{4n+4}.$$

The translation $x_2 = x'_2 + 1, x_3 = x'_3, x_4 = x'_4$ is now applied to the fundamental curves of the transformation and σ and κ are found to have $l_{2n+1} \equiv v'w - vw' = 0$ as common tangent curve at the point S . Moreover, it has been shown in §4 that the curve κ is tangent to each line γ_i at the point Γ_i .

The following table of characteristics and intersections may now be written:

$$\begin{aligned} \lambda &\sim (I)\phi : S^{2n+2}\Gamma_1, \dots, \Gamma_{4n+4}, \\ S &\sim (I)\sigma : S^{2n+1+(2n+1)t}\Gamma_1, \dots, \Gamma_{4n+4}, \\ \Gamma_i &\sim (I)\gamma_i : S\Gamma_i^{1+t}, \\ \kappa &\sim (I)\kappa : S^{2n+1+(2n+1)t}\Gamma_1^{1+t}, \dots, \Gamma_{4n+4}^{1+t}, \\ [\phi\phi] &: S^{4n^2+8n+4}\Gamma_1, \dots, \Gamma_{4n+4}F \text{ (the homaloidal web)}, \\ [\phi\sigma] &: S^{4n^2+6n+2}\Gamma_1, \dots, \Gamma_{4n+4}, \\ [\phi\gamma_i] &: S^{2n+2}\Gamma_i, [\sigma\gamma_i] : S^{2n+1}\Gamma_i, \\ [\phi\kappa] &: S^{4n^2+6n+2}\Gamma_i, \dots, \Gamma_{4n+4}K_1, \dots, K_{2n+3}, \\ [\sigma\kappa] &: S^{4n^2+4n+1+(2n+1)t}\Gamma_1, \dots, \Gamma_{4n+4}, \\ [\gamma_i\kappa] &: S^{2n+1}\Gamma_i^{1+t}. \end{aligned}$$

The F appearing above is the image of $[\lambda\lambda]$ and $K_1, \dots, K_{2n+3}$ are the intersections of κ and γ_i .

The jacobian is $\sigma^3\gamma$.

As the plane π generates the pencil on r , the I_{2n+3} generates the space $I_{2m+2n+5}$, whose equations may be obtained from (14) by replacing $u, u', v, v', w, w', (\alpha_{ijk}x)$ and β by $U, U', V, V', W, W', (c_{ijk}x)$ and x_1/x_2 , respectively.

When the surfaces are cut by a plane through s , a similar plane involution of order $2m + 3$ is found.

11. **The involutions associated with a special linear congruence.** If we take the directrices r and s of the congruence as

$$(15) \quad x_1 = x_4 = 0, \quad x_2 = x_3 = 0,$$

respectively, and pencils of surfaces F as in (2) and (3), where

$$u_i = \sum_{j=0}^m a_{ij} x_1^i x_4^{m-i}, \quad v_k = \sum_{j=0}^n b_{ik} x_2^i x_3^{n-i},$$

and so on, the equations of the involution, obtained by the methods of the preceding sections, are

$$(16) \quad I_{2m+2n+3} : \quad x_1 = Ry_1, \quad x_2 = Ry_2, \quad x_3 = Ry_3 + K, \quad x_4 = Ry_4,$$

where

$$\begin{aligned} R_{2m+2n+2} &= U'W - UW', & K_{2m+2n+3} &= U'V - UV', \\ V_{m+n+1} &= \sum_{i,k=0}^{m,n} u_i v_k (c_{ik,13} y_1 + c_{ik,23} y_2 + c_{ik,33} y_3 + c_{ik,43} y_4), \\ W_{m+n} &= \sum_{i,k=0}^{m,n} u_i v_k c_{ik,33}, \end{aligned}$$

and so on. Since the properties of such involutions are known, a complete discussion will not be given here.

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